

Magnetoelectric Cr_2O_3 and relativity theory

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Abstract. Relativity theory is useful for understanding the phenomenology of the magnetoelectric effect of the antiferromagnet chromium sesquioxide Cr_2O_3 in two respects: (i) One gets a clear idea about the physical dimensions of the electromagnetic quantities involved, in particular about the dimensions of the magnetoelectric moduli that we suggest to tabulate in future as dimensionless relative quantities; (ii) one can recognize and extract a temperature dependent, 4-dimensional pseudoscalar from the data of magnetoelectric experiments with Cr_2O_3 . This pseudoscalar piece of Cr_2O_3 is odd under time reflections and parity transformations and is structurally related (“isomorphic”) to the gyrator of electric network theory, the axion of particle physics, and the perfect electromagnetic conductor of electrical engineering.

PACS. 75.50.Ee Antiferromagnetics – 03.50.De Classical electromagnetism, Maxwell equations – 06.20.F- Units and standards – 46.05.+b General theory of continuum mechanics of solids – 14.80.Mz Axions and other Nambu-Goldstone bosons

1 Introduction

Magnetism is caused by moving electric charge, and the *motion* of an object is closely related to relativity theory; at least if the charge moves with a speed v that cannot be neglected as compared to the speed of light c . Accordingly, in the theory of magnetoelectricity that deals with materials where a magnetisation M is induced by an electric field E or a polarization P by a magnetic field B (or a magnetic excitation H), magnetic and electric fields are closely intertwined, and it is wise to put the electrodynamical theory in a relativistic form right away; and this all the more since Maxwell’s theory is known to be a relativistic theory covariant under all coordinate transformations (see Post [46]).

2 Physical dimensions in magnetoelectrics

2.1 Permittivity and permeability

Before one formulates a physical theory, in the case under consideration electrodynamics of material media, one has to make a basic decision: Do we want to formulate the laws in terms of “quantity equations” or in terms of

“numerical equations”? A quantity $A = \{A\}[A]$ is always a “numerical value” $\{A\}$ times a “dimension” $[A]$; and quantity equations interrelate these quantities. By their very definition quantity equations are valid in any system of units. Thus, if one puts the Maxwell equations in such a form, they are valid in SI (the international system of units), in the Gauss system of units, etc. By contrast, in the book of Jackson [29], for instance, on the top of each second page, it is indicated for which system of units (SI or Gaussian) the equations displayed are valid. Clearly, this distracts from the universal nature of the Maxwell equations—and we will not follow that convention. Also the elementary particle physicist prefer to formulate the laws in such a way that they are only valid in a certain system of units, with $\hbar = 1$, $c = 1$, etc.

In order to recognize the archetypal structure of the Maxwell equations, one should write them, as already Maxwell has done himself—who also discusses physical dimensions in quite some detail—in terms of quantity equations, see Post [46] or [18]. We will follow Post in this context.

In the past these matters of dimensions and units caused heated debates. One can get a good idea of this from the article of Brown [9]. He gives a good and complete description of how to change from one system of units to another one. However, in contrast to Post, he argues that choice of a dimension is more or less a convention. We disagree with this position. In our opinion the dimension of a quantity encodes the operational procedure of how to measure this quantity.

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The dimension of a quantity is an intrinsic and defining property of a quantity. We cannot, by suitable units, kill the dimension of a physical quantity. A dimension is something invariant. The electric excitation \mathbf{D} , for instance, can be measured in vacuum with the help of the Maxwellian double plates implying the dimension $[\mathbf{D}] = \text{charge/area} = q/\ell^2$, where q denotes the dimension of charge and ℓ the dimension of length. Similarly the electric field strength $[\mathbf{E}] = \text{force/charge} = (w/\ell)/q = (UIt/\ell)/(It) = U/\ell$, with w , U , I , and t as dimensions of work (energy), voltage, current, and time, respectively. Clearly then, quite independently of the system of units chosen, the permittivity defined by the ratio of the electric excitation and the electric field $\varepsilon = \mathbf{D}/\mathbf{E}$ has the intrinsic dimension $[\varepsilon] = [\mathbf{D}]/[\mathbf{E}] = (q/\ell^2)\ell/U = (It/\ell^2)\ell/U = 1/(Rv)$, where R and v are the dimensions of resistance and velocity, respectively. In particular, also in *vacuum* $[\varepsilon_0] = Y/v$, with $Y = 1/R$ as the dimension of admittance. Note that we talk about dimensional analysis, not about a system of units. Our discussion looks like one relating to SI; however, this is not the case—rather SI also uses quantity equations and hence some equations in SI resemble those we discuss here. In the Gaussian system, for example, one introduces effectively a new quantity $\mathbf{d} := \mathbf{D}/\varepsilon_0$ and thereby finds in vacuum $\mathbf{d} = \mathbf{E}$. However, \mathbf{d} carries no longer the dimension of an excitation and, accordingly, is only some superficial notational construct without operational interpretation. The conventional vacuum relation $\mathbf{D} = \mathbf{E}$, used in Gaussian units, defies a reasonable dimensional analysis.

For ordinary linear and isotropic materials we can write

$$\mathbf{D} = \varepsilon_r \varepsilon_0 \mathbf{E}, \quad \mathbf{H} = \frac{1}{\mu_r \mu_0} \mathbf{B}, \quad (1)$$

$$[\varepsilon_0] = \frac{Y}{v}, \quad [\mu_0] = \frac{1}{Yv}, \quad [\varepsilon_r] = [\mu_r] = 1,$$

where \mathbf{H} is the magnetic excitation and \mathbf{D} the electric excitation that are induced by the externally applied fields \mathbf{E} and \mathbf{B} . We immediately recognize that the moduli characterizing the material, namely the (relative) permittivity ε_r and the (relative) permeability μ_r are both dimensionless, quite independently of any system of units. This convenient parameterization is achieved as soon as we introduce some standard and constant ε_0 and μ_0 , with the dimensions of Y/v and $1/(Yv)$, respectively.

It turns out in a relativistic analysis of the Maxwell equations that an admittance—or resistance as its reciprocal—are four-dimensional *scalar* quantities even in the Riemannian spacetime of general relativity. This is why we concentrate on such a quantity with a dimension of admittance. That a velocity is inherent in electrodynamics is well-known and was already established by the experiment of Weber-Kohlrausch before the advent of Maxwell's theory, see [8]. This vacuum speed of light c is a *special-relativistic* 4-dimensional *scalar* as was uncovered when Minkowski set up a manifestly Poincaré-covariant version of electrodynamics (compare [17]). Therefore, two 4-dimensional scalar quantities, namely the *vacuum impedance* Y_0 and the *vacuum speed of light* c characterize the vacuum

from an electro-dynamical point of view. In different system of units they have different values, in SI, for instance, we have $Y_0 = \sqrt{\varepsilon_0/\mu_0} \approx 1/(377 \Omega)$ and $c = 1/\sqrt{\varepsilon_0\mu_0} \approx 3 \times 10^8 \text{ m/s}$. But the dimensionless permittivity/permeability, ε_r and μ_r , keep their values in all systems of units.

2.2 Four-dimensional excitation and field strength tensors

In order to extend this characterization of materials by dimensional quantities also to magnetoelectric moduli, we have first to turn to the 4-dimensional representation of electrodynamics. Each of the two sets (\mathbf{D}, \mathbf{H}) and (\mathbf{E}, \mathbf{B}) constitutes a relativistic “couple” that is, we can define 4-dimensional 2nd rank antisymmetric tensors according to $(\lambda, \nu = 0, 1, 2, 3)$

$$\mathfrak{G} = (\mathfrak{G}^{\lambda\nu}) = -(\mathfrak{G}^{\nu\lambda}) = \begin{pmatrix} 0 & D^1 & D^2 & D^3 \\ -D^1 & 0 & H_3 & -H_2 \\ -D^2 & -H_3 & 0 & H_1 \\ -D^3 & H_2 & -H_1 & 0 \end{pmatrix}, \quad (2)$$

$$F = (F_{\lambda\nu}) = -(F_{\nu\lambda}) = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B^3 & -B^2 \\ E_2 & -B^3 & 0 & B^1 \\ E_3 & B^2 & -B^1 & 0 \end{pmatrix}. \quad (3)$$

Even if one is not familiar with these expressions, we can see by a dimensional analysis that each couple is really intrinsically interrelated. We have $[\mathbf{H}] = I/\ell = q/(t\ell)$, $[\mathbf{D}] = q/\ell^2$, and $[\mathbf{D}] = (1/v)[\mathbf{H}]$. Accordingly, \mathbf{D} and \mathbf{H} are both *charge-related* quantities distinguished only by the dimension of velocity. This dimension of velocity mediates between time and space components like for a coordinate where $[x^0] = t$, $[x^1] = \ell = vt, \dots$, that is, $[x^0] = (1/v)[x^1], \dots$. Analogously, we have $[\mathfrak{G}^{01}] = [D^1] = (1/v)[H_1] = (1/v)[\mathfrak{G}^{23}]$ etc.

For the couple (\mathbf{E}, \mathbf{B}) we have, with ϕ as dimension of magnetic flux, $[\mathbf{E}] = U/\ell = \phi/(t\ell)$ and $[\mathbf{B}] = \phi/\ell^2$, that is, $[\mathbf{E}] = v[\mathbf{B}]$. In other words, \mathbf{E} and \mathbf{B} are *flux-related* (or, via the Lorentz force, force-related) quantities.

Apparently, the matrices in (2) and (3) can be consistently set up from a dimensional point of view, including the factors involving the velocity, provided the mentioned couples are used. Thus, $[\mathfrak{G}] \sim \text{charge}$, $[F] \sim \text{flux}$, and $[\mathfrak{G}] \times [F] \sim \text{charge} \times \text{flux} = \text{action}$; for a detailed discussion see [18]. We recognize thereby that the scalar (density) with the dimension of an action,

$$\frac{1}{4} \mathfrak{G}^{\lambda\nu} F_{\lambda\nu} = \frac{1}{2} (\mathbf{B} \cdot \mathbf{H} - \mathbf{D} \cdot \mathbf{E}), \quad (4)$$

must have a fundamental importance in electrodynamics: It represents the Lagrangian of the electromagnetic field and is equally well a 4-dimensional general relativistic scalar.

In the literature these results are hardly recognized. Landau-Lifshitz [34], for example, like most of the engineers, see Sihvola and Lindell [59], take the bastard pair (\mathbf{E}, \mathbf{H}) for their considerations. From a dimensional as

well as from a relativistic point of view, there is no legitimacy to give birth to such a hybrid. All dimensions in relations get mixed up by such an unholy pair.¹

2.3 Dimensionless magnetoelectric moduli

According to (2) and (3), the general *local* and *linear* constitutive law is expected to be of the general form $\mathfrak{G} \propto F$. The simplest case is that of an isotropic medium. If then, in the magnetoelectric case, a magnetic field \mathbf{B} (or an electric field \mathbf{E}) is applied, the additional response will be an electric excitation \mathbf{D} (or a magnetic excitation \mathbf{H}):

$$\mathbf{D} = \alpha_r \alpha_0 \mathbf{B}, \quad \mathbf{H} = \beta_r \beta_0 \mathbf{E}; \quad (5)$$

$$[\alpha_0] = [\beta_0] = \frac{[\mathbf{D}]}{[\mathbf{B}]} = \frac{[\mathbf{H}]}{[\mathbf{E}]} = Y, \quad [\alpha_r] = [\beta_r] = 1.$$

Since the dimensions of α_0 and β_0 are the same, we can put them equally to the vacuum admittance: $\alpha_0 = \beta_0 = Y_0$. If we allow an electric and a magnetic field to be present at the same time, then (1) combined with (5) for isotropic media leads to the ansatz

$$\mathbf{D} = Y_0 \left(\varepsilon_r \frac{1}{c} \mathbf{E} + \alpha_r \mathbf{B} \right), \quad (6)$$

$$\mathbf{H} = Y_0 \left(\frac{1}{\mu_r} c \mathbf{B} + \beta_r \mathbf{E} \right); \quad (7)$$

$$[Y_0] = \text{admittance}, \quad [c] = \text{velocity},$$

$$[\varepsilon_r] = [\mu_r] = [\alpha_r] = [\beta_r] = 1.$$

Note that $Y_0/c = \varepsilon_0$ and $Y_0 c = 1/\mu_0$.

Hence we see that all electric, magnetic, and magnetoelectric moduli, independent of any system of units, can be described by the *dimensionless* quantities ε_r , μ_r , α_r , β_r . This has been shown, amongst others, by Post [46], O'Dell [44], Ascher [1,2,3], and Hehl et al. [21]. Units such as those used in Ref.[16], p. R128, for example, namely (V/cm)/Oe should be abandoned. They complicate the comparison of different experiments.

If experimentalists collect magnetoelectric constants, then our suggestion is to *tabulate them in their dimensionless relative form*, since these values are totally independent of the system of units used and represent their genuine values.

2.4 Local and linear magnetoelectric constitutive law

Now that we have an elementary understanding of the dimensions of the magnetoelectric moduli, we proceed to

¹ There is one situation, however, in which the non-relativistic pair (\mathbf{E}, \mathbf{H}) can be useful. Since in the Maxwell equations only \mathbf{D} and \mathbf{B} enter as time derivatives, see $\text{curl } \mathbf{H} = j + \dot{\mathbf{D}}$ and $\text{curl } \mathbf{E} + \dot{\mathbf{B}} = 0$, the pair (\mathbf{D}, \mathbf{B}) and by implication (\mathbf{E}, \mathbf{H}) are useful for formulating the initial value problem in electrodynamics, as was pointed out by Perlick [45]. In this situation one has to leave the manifestly relativistic covariant formulation of electrodynamics and introduces necessarily a particular slicing of the 4-dimensional spacetime that destroys the manifest covariance of the theory.

formulate, following Tamm [63], Post [46], and our discussion in [21], a 4-dimensional covariant law for a local, linear, and anisotropic medium thereby generalizing the corresponding isotropic law (6),(7),

$$\mathfrak{G}^{\lambda\nu} = \frac{1}{2} \chi^{\lambda\nu\sigma\kappa} F_{\sigma\kappa} = \frac{1}{2} Y_0 \xi^{\lambda\nu\sigma\kappa} F_{\sigma\kappa}, \quad (8)$$

where the constitutive tensor χ of rank 4 and weight +1 is of dimension $[\chi] = Y$; thus, $[\xi] = 1$. Because of the antisymmetry of $\mathfrak{G}^{\lambda\nu}$ and $F_{\sigma\kappa}$, we have $\chi^{\lambda\nu\sigma\kappa} = -\chi^{\lambda\nu\kappa\sigma} = -\chi^{\nu\lambda\sigma\kappa}$. An antisymmetric pair of indices corresponds, in four dimensions, to six independent components. Thus, the constitutive tensor can be considered as a 6×6 matrix with 36 independent components.

A 6×6 matrix can be decomposed in its tracefree symmetric part (20 independent components), its antisymmetric part (15 components), and its trace (1 component). On the level of $\chi^{\lambda\nu\sigma\kappa}$, this *decomposition* is reflected in

$$\chi^{\lambda\nu\sigma\kappa} = {}^{(1)}\chi^{\lambda\nu\sigma\kappa} + {}^{(2)}\chi^{\lambda\nu\sigma\kappa} + {}^{(3)}\chi^{\lambda\nu\sigma\kappa}. \quad (9)$$

$$36 = 20 \oplus 15 \oplus 1.$$

The third part, the *axion* part, is totally antisymmetric and as such proportional to the totally antisymmetric Levi-Civita symbol, ${}^{(3)}\chi^{\lambda\nu\sigma\kappa} := \tilde{\alpha} \tilde{\epsilon}^{\lambda\nu\sigma\kappa}$. We remind ourselves that the Levi-Civita symbol $\tilde{\epsilon}^{\lambda\nu\sigma\kappa} = +1$ or -1 depending whether $\lambda\nu\sigma\kappa$ denotes an even or an odd permutation of the numbers 0123, respectively; it is zero otherwise, see [61]. We denote pseudoscalars with a tilde.

The second part, the *skewon* part, is defined according to ${}^{(2)}\chi^{\mu\nu\lambda\rho} := \frac{1}{2}(\chi^{\mu\nu\lambda\rho} - \chi^{\lambda\rho\mu\nu})$. If the constitutive equation can be derived from a Lagrangian, which is the case as long as only reversible processes are considered, then ${}^{(2)}\chi^{\lambda\nu\sigma\kappa} = 0$. The *principal* part ${}^{(1)}\chi^{\lambda\nu\sigma\kappa}$ fulfills the symmetries ${}^{(1)}\chi^{\lambda\nu\sigma\kappa} = {}^{(1)}\chi^{\sigma\kappa\lambda\nu}$ and ${}^{(1)}\chi^{[\lambda\nu\sigma\kappa]} = 0$. The constitutive relation now reads

$$\mathfrak{G}^{\lambda\nu} = \frac{1}{2} \left({}^{(1)}\chi^{\lambda\nu\sigma\kappa} + {}^{(2)}\chi^{\lambda\nu\sigma\kappa} + \tilde{\alpha} \tilde{\epsilon}^{\lambda\nu\sigma\kappa} \right) F_{\sigma\kappa}$$

$$= \frac{1}{2} Y_0 \left({}^{(1)}\xi^{\lambda\nu\sigma\kappa} + {}^{(2)}\xi^{\lambda\nu\sigma\kappa} + \tilde{\alpha}_r \tilde{\epsilon}^{\lambda\nu\sigma\kappa} \right) F_{\sigma\kappa}, \quad (10)$$

with $\tilde{\alpha}_r$ as dimensionless, $[\tilde{\alpha}_r] = 1$.

In order to compare (10) with experiments—the law (6),(7) is a special case of it—we have to split (10) into time and space parts. As shown in [20] in detail, we can parameterize the *principal* part by the 6 permittivities $\varepsilon^{ab} = \varepsilon^{ba}$, the 6 permeabilities $\mu_{ab} = \mu_{ba}$, and the 8 magnetoelectric pieces γ^a_b (its trace vanishes, $\gamma^c_c = 0$, summation over c !) and the *skewon* part by the 3 permittivities n_a , the 3 permeabilities m^a , and the 9 magnetoelectric pieces s_a^b . Then (10) can be rewritten as $(a, b, c = 1, 2, 3)$

$$D^a = (\varepsilon^{ab} - \epsilon^{abc} n_c) E_b$$

$$+ (\gamma^a_b + s_b^a - \delta_b^a s_c^c) B^b + \tilde{\alpha} B^a, \quad (11)$$

$$H_a = (\mu_{ab}^{-1} - \epsilon_{abc} m^c) B^b$$

$$+ (-\gamma^b_a + s_a^b - \delta_a^b s_c^c) E_b - \tilde{\alpha} E_a. \quad (12)$$

Here $\epsilon^{abc} = \hat{\epsilon}_{abc} = \pm 1, 0$ are the 3-dimensional Levi-Civita symbols. As can be seen from our derivation, $\tilde{\alpha} = \tilde{\alpha}_r Y_0$ is a

4-dimensional pseudo- (or axial) scalar, whereas s_c^c is only a 3-dimensional scalar. The cross-term γ^a_b is related to the Fresnel-Fizeau effects. The skewon contributions m^c, n_c are responsible for electric and magnetic Faraday effects, respectively, whereas the skewon terms s_a^b describe optical activity. Equivalent constitutive relations were formulated, amongst others, by Serdyukov et al. [55], by Lindell [35], and by Spaldin et al. [62]. Magnetoelectric effects in spiral magnets and in ferromagnets were recently studied by Mostovoy [39] and Dzyaloshinskii [14], respectively.

From (11),(12), for $\varepsilon^{ab} = \varepsilon g^{ab}$, where g^{ab} are the components of the 3-dimensional (contravariant) metric, $(\mu^{-a})_{ab} = (\mu^{-1})g_{ab}$, and all other magnetoelectric moduli vanishing, we recover the special case

$$D^a = \varepsilon E^a + \tilde{\alpha} B^a, \quad (13)$$

$$H_a = (\mu^{-1})B_a - \tilde{\alpha} E_a, \quad (14)$$

with $E^a = g^{ab}E_b$ and $B_a = g_{ab}B^b$. This is the isotropic case of (6),(7); but additionally, we recognize that in (7) we have to require

$$\beta_r = -\alpha_r \quad (15)$$

for consistency.

Incidentally, if necessary, the linear law (8) can be amended with a *quadratic* term according to the scheme

$$\Theta^{\lambda\nu} = \frac{1}{2}\chi^{\lambda\nu\sigma\kappa}F_{\sigma\kappa} + \frac{1}{4}\eta^{\lambda\nu\sigma\kappa\mu\tau}F_{\sigma\kappa}F_{\mu\tau}, \quad (16)$$

with an additional constitutive tensor η of 126 independent components. In the reversible case they reduce to 56 independent components.

3 The relativistic pseudoscalar $\tilde{\alpha}$

3.1 Violation of the Post constraint

According to some hand-waving arguments of Post, see [46], p. 129, the pseudoscalar $\tilde{\alpha}$ in (11), (12) (and even its first derivative) should vanish for the vacuum as well as for all media. Lakhtakia, see [32], dubbed the relation $\tilde{\alpha} = 0$ the “Post constraint” and pushed it as a presumed consequence of Maxwell’s equations quite vigorously. Lakhtakia’s arguments have never been convincing to a determined minority, see, for instance, Sihvola and Tretyakov [56,60]. More recently Lakhtakia [33] admitted that he was driven to the Post constraint by the following two prejudices [33]:

“(i) The idea of a nonreciprocal but isotropic medium appears oxymoronic [contradictory].”

“(ii) The idea of a constitutive parameter that vanishes from the macroscopic Maxwell equations for a linear medium but leaves behind its spatiotemporal derivatives appears bizarre.”

To (i): This is the case of (13),(14). If a material is “charged” with the 4-dimensional *pseudoscalar* $\tilde{\alpha}$ —admittedly a case that is rare in nature—then an orientation dependence of $\tilde{\alpha}$ is implied; why should this contradict the 3-dimensional isotropy of the medium?

To (ii): If $\tilde{\alpha}$ is spacetime dependent, $\tilde{\alpha} = \tilde{\alpha}(x)$, then it emerges in the field equations of electrodynamics, as had already been shown by Wilczek [70] in his axion-electrodynamics. In the constant case, the $\tilde{\alpha}$ jumps at the interface between the medium and vacuum and yields a contribution to the corresponding boundary conditions, see [41]. Moreover, at least since the very exact measurements of Wiegmann et al. [68] no doubt was possible at the violation of the Post constraint in the case of the antiferromagnet Cr_2O_3 .

3.2 Dzyaloshinskii’s theory for the magnetoelectric effect of the antiferromagnet Cr_2O_3

Landau and Lifshitz (1956, see the first Russian edition of [34]) predicted the magnetoelectric effect as a phenomenon that “results from a linear relation between the magnetic and the electric fields in a substance, which would cause, for example, a magnetization proportional to the electric field...” It “can occur for certain magnetic crystal symmetry classes.” Dzyaloshinskii [11], on the basis of neutron scattering data and susceptibility measurements, found out that the antiferromagnetic chromium sesquioxide Cr_2O_3 has the desired magnetic symmetry class. Starting from a thermodynamic potential quadratic and bilinear in \mathbf{E} and \mathbf{H} , he developed a theory for the magnetoelectric constitutive relations for Cr_2O_3 . Written as quantity equations, they are valid in an arbitrary system of units and read

$$D_x = \varepsilon_{\perp} \frac{Y_0}{c} E_x + \alpha_{\perp} \frac{1}{c} H_x, \quad (17)$$

$$D_y = \varepsilon_{\perp} \frac{Y_0}{c} E_y + \alpha_{\perp} \frac{1}{c} H_y, \quad (18)$$

$$D_z = \varepsilon_{\parallel} \frac{Y_0}{c} E_z + \alpha_{\parallel} \frac{1}{c} H_z \quad (19)$$

and

$$B_x = \mu_{\perp} \frac{1}{Y_0 c} H_x + \alpha_{\perp} \frac{1}{c} E_x, \quad (20)$$

$$B_y = \mu_{\perp} \frac{1}{Y_0 c} H_y + \alpha_{\perp} \frac{1}{c} E_y, \quad (21)$$

$$B_z = \mu_{\parallel} \frac{1}{Y_0 c} H_z + \alpha_{\parallel} \frac{1}{c} E_z. \quad (22)$$

The z -axis is parallel to the optical axis of Cr_2O_3 . Here we have (relative) permittivities parallel and perpendicular to the z -axis of the crystal, namely $\varepsilon_{\parallel}, \varepsilon_{\perp}$, analogous (relative) permeabilities $\mu_{\parallel}, \mu_{\perp}$ and magnetoelectric moduli $\alpha_{\parallel}, \alpha_{\perp}$.

As we can see from (11) and (12), we have to get (\mathbf{D}, \mathbf{H}) on the left hand side and (\mathbf{E}, \mathbf{B}) on the right hand side in order to end up with a constitutive law that is

written in a relativistically covariant form. We find,

$$D_x = Y_0 \left[\left(\varepsilon_{\perp} - \frac{\alpha_{\perp}^2}{\mu_{\perp}} \right) \frac{1}{c} E_x + \frac{\alpha_{\perp}}{\mu_{\perp}} B_x \right], \quad (23)$$

$$D_y = Y_0 \left[\left(\varepsilon_{\perp} - \frac{\alpha_{\perp}^2}{\mu_{\perp}} \right) \frac{1}{c} E_y + \frac{\alpha_{\perp}}{\mu_{\perp}} B_y \right], \quad (24)$$

$$D_z = Y_0 \left[\left(\varepsilon_{\parallel} - \frac{\alpha_{\parallel}^2}{\mu_{\parallel}} \right) \frac{1}{c} E_z + \frac{\alpha_{\parallel}}{\mu_{\parallel}} B_z \right] \quad (25)$$

and

$$H_x = Y_0 \left[\frac{1}{\mu_{\perp}} c B_x - \frac{\alpha_{\perp}}{\mu_{\perp}} E_x \right], \quad (26)$$

$$H_y = Y_0 \left[\frac{1}{\mu_{\perp}} c B_y - \frac{\alpha_{\perp}}{\mu_{\perp}} E_y \right], \quad (27)$$

$$H_z = Y_0 \left[\frac{1}{\mu_{\parallel}} c B_z - \frac{\alpha_{\parallel}}{\mu_{\parallel}} E_z \right]. \quad (28)$$

We can compare these equations with (11) and (12). Since Dzyaloshinskii assumed reversibility, the skewon piece with its 15 independent components vanishes identically. Hence (11) and (12) reduce to

$$D^a = \varepsilon^{ab} E_b + \gamma^a_b B^b + \tilde{\alpha} B^a, \quad (29)$$

$$H_a = \mu_{ab}^{-1} B^b - \gamma^b_a E_b - \tilde{\alpha} E_a, \quad (30)$$

with 21 independent moduli. The 4-dimensional pseudoscalar $\tilde{\alpha}$ (alternatively called the axion parameter) represents 1 component. We compare (29) and (30) with (23) to (28) and note that Dzyaloshinskii used Cartesian coordinates such that $B^x = B_x$ etc. Then we find the permittivity

$$\varepsilon^{ab} = \frac{Y_0}{c} \begin{pmatrix} \varepsilon_{\perp} - \frac{\alpha_{\perp}^2}{\mu_{\perp}} & 0 & 0 \\ 0 & \varepsilon_{\perp} - \frac{\alpha_{\perp}^2}{\mu_{\perp}} & 0 \\ 0 & 0 & \varepsilon_{\parallel} - \frac{\alpha_{\parallel}^2}{\mu_{\parallel}} \end{pmatrix}, \quad (31)$$

the impermeability

$$\mu_{ab}^{-1} = Y_0 c \begin{pmatrix} \mu_{\perp}^{-1} & 0 & 0 \\ 0 & \mu_{\perp}^{-1} & 0 \\ 0 & 0 & \mu_{\parallel}^{-1} \end{pmatrix}, \quad (32)$$

the magnetoelectric γ matrix

$$\gamma^a_b = \frac{1}{3} \left(\frac{\alpha_{\perp}}{\mu_{\perp}} - \frac{\alpha_{\parallel}}{\mu_{\parallel}} \right) Y_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad (33)$$

and the pseudoscalar (or axion) piece

$$\tilde{\alpha} = \frac{1}{3} \left(2 \frac{\alpha_{\perp}}{\mu_{\perp}} + \frac{\alpha_{\parallel}}{\mu_{\parallel}} \right) Y_0. \quad (34)$$

Conventionally, in the magnetoelectric literature the γ -matrix and $\tilde{\alpha}$ are collected in the “relativistic” matrix

$${}^{\text{rel}}\alpha^a_b := \gamma^a_b + \tilde{\alpha} \delta^a_b = Y_0 \begin{pmatrix} \frac{\alpha_{\perp}}{\mu_{\perp}} & 0 & 0 \\ 0 & \frac{\alpha_{\perp}}{\mu_{\perp}} & 0 \\ 0 & 0 & \frac{\alpha_{\parallel}}{\mu_{\parallel}} \end{pmatrix}. \quad (35)$$

We called it relativistic, since it occurs in the context of the relativistic (\mathbf{E}, \mathbf{B}) system. Later, the general form of these matrices for the crystal structure of Cr₂O₃ was confirmed by O’Dell [44], amongst others. Note that $\varepsilon_{\parallel}, \varepsilon_{\perp}, \mu_{\parallel}, \mu_{\perp}, \alpha_{\parallel}$, and α_{\perp} , according to their definitions (17) to (22), are measured in an external \mathbf{E} and/or an external \mathbf{H} field.

3.3 Experiments of Astrov, Rado & Folen, and Wiegmann et al.

The magnetoelectric effect for Cr₂O₃ was first found experimentally by Astrov [4] for the *electrically* induced magnetoelectric effect (called ME_E in future) and by Rado & Folen [47] for the *magnetically* induced magnetoelectric effect (ME_H). In both investigations single crystals of Cr₂O₃ were used. In the ME_E experiments [4, 47], Eqs.(20) to (22) were verified (H switched off) and in the ME_H experiments [47] Eqs.(17) to (19) (E switched off). In particular, Rado & Folen made both type of experiments and found that the magnetoelectric moduli α_{\perp} and α_{\parallel} for ME_E experiments coincide with those of the ME_H experiments. This proves the vanishing of the skewon part ${}^{(2)}\chi^{\lambda\nu\sigma\kappa}$ of the constitutive tensor $\chi^{\lambda\nu\sigma\kappa}$ for Cr₂O₃.

Accordingly, these experiments confirmed Dzyaloshinskii’s theory for Cr₂O₃ below the spin-flop phase. Further experiments were then done mainly for the ME_H case. Particular accurate measurements are those of Wiegmann et al. [68], see also [67, 69]. They took magnetic fields B as high as 20 tesla and measured from liquid Helium up to room temperature. Wiegmann et al. took a quasi-static magnetic field and thereby proved explicitly that measurements with magnetic fields of some kilo hertz can be extrapolated to static measurements. The sign of $\alpha_{\perp}(T)$ relative to $\alpha_{\parallel}(T)$ was left open in [68]. Hence we took that from Astrov [4].

We plotted the values of $\alpha_{\perp}(T)$ and $\alpha_{\parallel}(T)$ from the references quoted. We have shown in [21] that the permeabilities $\mu_{\perp} \approx \mu_{\parallel} \approx 1$. Then we can determine $\tilde{\alpha}$ by means of (34). The values we found [21] are displayed in Fig. 1. Up to about 163 K, the pseudoscalar $\tilde{\alpha}$ is negative, for higher temperatures positive until it vanishes at the Néel temperature of about 308 K. Its maximal value we find at 285 K:

$$\tilde{\alpha}_{\text{max}} \text{ (at 285 K)} \approx 3.10 \times 10^{-4} Y_0. \quad (36)$$

Consequently, we have definitely a nonvanishing pseudoscalar $\tilde{\alpha} = \tilde{\alpha}_r Y_0$ with a maximal $\tilde{\alpha}_r$ of the order of 10^{-4} . It is a small effect, but it does exist.

3.4 New predictions

3.4.1 Search for the first *cubic* magnetoelectric crystal

As we saw in Sec. 3.3, the parallel and perpendicular permeabilities for Cr₂O₃ are approximately equal, $\mu_{\parallel} \approx \mu_{\perp} = \mu$. Then the magnetoelectric γ matrix of (33) becomes

$$\gamma^a_b \approx \frac{Y_0}{3} \frac{\alpha_{\perp} - \alpha_{\parallel}}{\mu_r} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \quad (37)$$

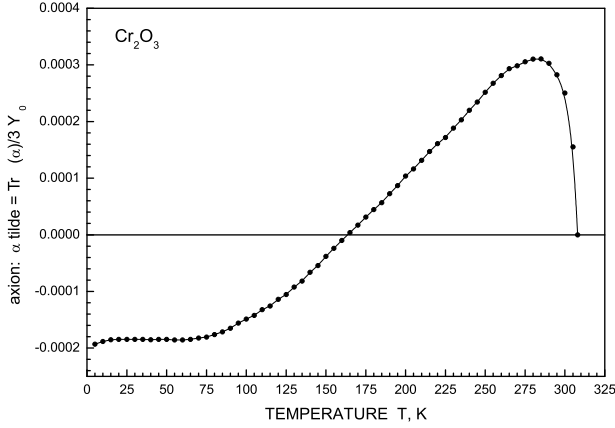


Fig. 1. The pseudoscalar (or axion) piece $\tilde{\alpha}$ of Eq. (34) of Cr_2O_3 in units of the vacuum admittance Y_0 as a function of the temperature T in kelvin; in SI, $Y_0 \approx 1/(377 \text{ ohm})$; for details see [21].

The question is now: Can we find a substance in which

$$\alpha_{\perp} = \alpha_{\parallel}, \quad (38)$$

that is, in which the matrix γ^{ab} vanishes *for all temperatures*? This would be interesting to know since then one would have a substance, like in (13) and (14), in which the *only* magnetoelectric piece would be the pseudoscalar (or axion) piece $\tilde{\alpha}$.

We can take from Rivera [48], Fig.2, box 16, that three equal entries in the diagonal of the matrix (37) are only possible in the following five antiferromagnetic *cubic* point groups:

$$23, m\bar{3}', 432, \bar{4}3m', m'\bar{3}m'. \quad (39)$$

Schmid [53], p.23, has explained that the *Cr-Br- and Cr-I-boracites* are antiferromagnetic and cubic below 17 K and 54 K, respectively. The Cu-I boracite stays also cubic down to very low temperatures [7] and is expected to behave analogously to Cr-I boracite, since both Cr^{++} and Cu^{++} are Jahn-Teller ions. However, the Néel temperature of Cu-I boracite is not known so far. Due to the cubic paramagnetic prototype point group $\bar{4}3m1'$, only two antiferromagnetic point groups are allowed:

$$\bar{4}3m' \text{ and } \bar{4}3m. \quad (40)$$

Whereas the point group $\bar{4}3m'$ allows the linear magnetoelectric (ME) effect (invariant $\bar{\alpha}_{ab}E_aH_b$) and both higher order ME effects (invariants $\bar{\gamma}_{abc}H_aE_bE_c$ and $\bar{\beta}_{abc}E_aH_bH_c$), the point group $\bar{4}3m$ allows only the invariant $\bar{\beta}_{abc}E_aH_bH_c$ (see [52], Table 2). It could be that *one or more of these three boracites have $\bar{4}3m'$ symmetry*.

In order to check this, ME measurements on single crystals would be desirable. However, the linear ME effect can in principle also be detected on compacted, magnetoelectrically annealed, polycrystalline powders or ceramics [22,23,24], both by ME_H and ME_E methods. The

$\bar{\beta}_{abc}E_aH_bH_c$ -effect cancels out statistically in powders and ceramics because the sign of the coefficient is rigorously linked to the orientation of the non-centrosymmetric, nuclear absolute structure (note, only in poled ferroelectric ceramics this statistics is broken and the effect becomes measurable in principle, when using the ME_H effect). Because the $\bar{\gamma}_{abc}H_aE_bE_c$ -effect cannot be measured by a ME_H quasistatic method, it would not disturb when one measures the linear effect of a phase of the point group $\bar{4}3m'$. The $\bar{\gamma}_{abc}H_aE_bE_c$ -effect can be measured by a ME_E method or by a ME_H technique in presence of a static electric field, as first shown by O'Dell [43]. As a consequence, even ME measurements on compacted powders or on ceramics would allow to distinguish between the two point groups, with $\bar{4}3m$ showing no ME effect at all and with $\bar{4}3m'$ only the linear effect when a ME_H quasistatic method is used. We would like to suggest to grow single crystals [51,54] and to perform ME_H measurements [49] both on single crystals and polycrystalline samples. A polycrystalline probe of a cubic magnetoelectric crystal is obviously completely isotropic for the linear magnetoelectric effect. Hence with a view on applications, the production of polycrystalline material could be simpler than growing single crystals.

3.4.2 External magnetic *octupole* field of a cubic magnetoelectric crystal

Dzyaloshinskii [13] hypothesized that there may exist an external magnetic field for free cubic (or axion type) magnetoelectric media. Normally, the external magnetic field outside an antiferromagnet decays exponentially. Much earlier, Dzyaloshinskii [12] predicted the existence of an external magnetic and electric field for magnetoelectric media with high symmetry, and specifically for the antiferromagnetic Cr_2O_3 . Such a field decreases according to a power law from the surface of a body, in contrast to the exponential decay typical for substances with lower symmetries. For Cr_2O_3 , this external magnetic field behaves as the field of a magnetic *quadrupole*, which was subsequently confirmed experimentally by Astrov et al. [5,6].

For the magnetoelectric crystal with a cubic symmetry, which is characterized by the purely axion piece in the linear constitutive relation, we expect the similar effect [13]. Taking, for definiteness, a ball made of such a material (recall that Astrov et al. in the 1960s used little balls made out of the antiferromagnetic Cr_2O_3), one can perform a usual expansion of the electric and magnetic fields into spherical harmonics. From a preliminary qualitative analysis of the surface charges and currents induced, one can expect an external magnetic field of the *octupole* structure, namely,

$$B \sim -\tilde{\alpha} \nabla \left(\frac{\mathcal{Y}_{4m}}{r^5} \right), \quad (41)$$

which is directly proportional to the axion piece $\tilde{\alpha}$ (here, as usual, the $\mathcal{Y}_{lm}(\theta, \varphi)$ denote the spherical harmonics). This preliminary result is based on the fact that \mathcal{Y}_{4m} is

invariant under the cubic symmetry group of the prospective cubic axion material.

3.5 Relations of $\tilde{\alpha}$ to gyrator, axion, and perfect electromagnetic conductor (PEMC)

The constitutive law for the pseudoscalar piece, namely (5) together with (15), reads

$$\begin{aligned} D^a &= \tilde{\alpha} B^a, \\ H_a &= -\tilde{\alpha} E_a, \end{aligned} \quad (42)$$

see also (13), (14). This structure is not unprecedented, see the discussion in [20]. In electrical engineering, in the theory linear networks, more specifically in the theory of two ports (or four poles), Tellegen [64,65] invented the *gyrator* the defining relations of which are

$$\begin{aligned} v_1 &= -s i_2, \\ v_2 &= s i_1, \end{aligned} \quad (43)$$

where v are voltages and i currents of the ports 1 and 2, respectively, see also [44]. Let us quote from Tellegen [65], p.189: “The ideal gyrator has the property of ‘gyrating’ a current into a voltage, and vice versa. The coefficient s , which has the dimension of a resistance, we call the gyration resistance; $1/s$ we call the gyration conductance.” The gyrator is a nonreciprocal network element that is discussed in the electrical engineering literature [15,31], for more recent developments see, e.g., [50] and [71].

For dimensional reasons, the electromagnetic excitations (D^a, H_a) are related to the currents and the field strengths (E_a, B^a) to the voltages, that is,

$$\begin{aligned} E_a &= -s H_a, \\ B^a &= s D^a. \end{aligned} \quad (44)$$

If we put $s = 1/\tilde{\alpha}$, then (44) and (42) coincide. Similar as the gyrator rotates currents into voltages, the axion piece ‘rotates’ the excitations, modulo a resistance, into the field strengths.

The next “isomorphic” structure to discuss is *axion electrodynamics*, see Ni [40], Wilczek [70], and, for more recent work, Itin [26,27,28]. We add to the usual Maxwell-Lorentz law for vacuum electrodynamics $\mathfrak{G}^{\lambda\nu} = Y_0 \sqrt{-g} F^{\lambda\nu}$ an axion piece patterned after the last term in (10), then we have the constitutive law for axion electrodynamics,

$$\mathfrak{G}^{\lambda\nu} = Y_0 \sqrt{-g} F^{\lambda\nu} + \frac{1}{2} \tilde{\alpha} \tilde{\epsilon}^{\lambda\nu\sigma\kappa} F_{\sigma\kappa}. \quad (45)$$

In order to get a feeling for the Lagrangian of axion electrodynamics, we substitute (13), (14), and (15) into the right-hand-side of (4); after some algebra we find

$$\begin{aligned} \frac{1}{4} \mathfrak{G}^{\lambda\nu} F_{\lambda\nu} &= \frac{1}{2} (\mathbf{B} \cdot \mathbf{H} - \mathbf{D} \cdot \mathbf{E}) \\ &= \underbrace{\frac{1}{2\mu_r\mu_0} \mathbf{B}^2 - \frac{1}{2\epsilon_r\epsilon_0} \mathbf{E}^2}_{\text{Maxwell-Lorentz part}} - \underbrace{\tilde{\alpha}_r Y_0 \mathbf{E} \cdot \mathbf{B}}_{\text{axion part}}. \end{aligned} \quad (46)$$

As we saw, in Cr_2O_3 we had $\tilde{\alpha}_r \approx 10^{-4}$. It is everybody’s guess what it could be in elementary particle theory. There, in spite of extended searches via the coupling of photons to the hypothetical pseudoscalar axion field, amongst other methods, nothing has been found so far, see the review of Carosi et al. [10].

These isomorphisms even found a further field of applications: In 2005, Lindell & Sihvola [36,37], see also [35], introduced the new concept of a *perfect electromagnetic conductor* (PEMC). It also obeys the constitutive law (42). Some applications were studied by Jancewicz [30] and by Illahi and Naqvi [25], see also the references given therein. The PEMC is a generalization of the perfect electric and the perfect magnetic conductor. In this sense, it is the ‘ideal’ electromagnetic conductor that can be hopefully built by means of a suitable *metamaterial*, see Sihvola [57]. The pseudoscalar $\tilde{\alpha}$ is called Tellegen parameter by Lindell et al., see [38], p.13 (for a more general view, see [58]); artificial Tellegen material has been produced and positively tested by Tretyakov et al. [66], amongst others.

Accordingly, (i) the gyrator, (ii) the axion field of axion electrodynamics, (iii) the pseudoscalar $\tilde{\alpha}$ of Cr_2O_3 , and (iv) the PEMC are isomorphic structures. Wilczek (private communication) remarked to these isomorphisms between these four systems: “It’s a nice demonstration of the unity of physics.” Already in his paper of 1987 [70] he argued that “...it is...not beyond the realm of possibility that fields whose properties partially mimic those of axion fields can be realized in condensed-matter systems.” ...This is, indeed, the case: The pseudoscalar $\tilde{\alpha}$ of Cr_2O_3 is the structure Wilczek looked for.

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